

DERIVATION OF LOCAL GAUGE FREEDOM FROM A MEASUREMENT PRINCIPLE

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We define operator manifolds as manifolds on which a spectral measure on a Hilbert space is given as additional structure. The spectral measure mathematically describes space as a quantum mechanical observable. We show that the vectors of the Hilbert space can be represented as functions on the manifold. The arbitrariness of this representation is interpreted as local gauge freedom. In this way, the physical gauge principle is linked with quantum mechanical measurements of the position variable. We derive the restriction for the local gauge group to be $U(m)$, where m is the number of components of the wave functions.

1. Operator Manifolds

In nonrelativistic quantum mechanics, the physical observables \mathcal{O} are described by self-adjoint operators on a Hilbert space H . The wave function of a particle is a vector $\Psi \in H$; measurements correspond to the calculation of expectation values $\langle \Psi | \mathcal{O} | \Psi \rangle$. In this paper, we will concentrate on the observables for space. These observables play a special role because they determine the geometry of the physical system. Usually they are given by mutually commuting operators $(X^i)_{i=1,\dots,3}$ with a continuous spectrum (i.e. the multiplication operators with the coordinate functions in position space). Since the X^i depend on the choice of the coordinate system in \mathbb{R}^3 , it is more convenient to consider their spectral measure $(E_x)_{x \in \mathbb{R}^3}$ (for basic definitions see e.g. ¹). The operators X^i can be reconstructed from the spectral measure by integrating over the coordinate functions,

$$X^i = \int_{\mathbb{R}^3} x^i dE_x \quad . \quad (1)$$

We want to study this functional analytic point of view in the more general setting that space is a manifold.

Definition 1 Let M be a manifold of dimension n , (μ, \mathcal{M}) a positive, σ -finite measure on M with σ -algebra \mathcal{M} . Furthermore, let H be a separable Hilbert space,

$$E : \mathcal{M} \rightarrow P(H)$$

a spectral measure ($P(H)$ denotes the projection operators of H), which is absolutely continuous with respect to μ , $E \ll \mu$ (for basic measure theory see e.g. ²). (M, μ, H, E) is called **operator manifold**.

For simplicity, the reader may think of $d\mu$ as the measure $\sqrt{g} d^n x$ on a Riemannian manifold and of \mathcal{M} as the Lebesgue measurable sets. The requirement $E \ll \mu$ is mainly a technical simplification.

Example 1 1. For a scalar particle, we choose $M = \mathbb{R}^3$, $H = L^2(M)$, and (μ, \mathcal{M}) the Lebesgue measure. We define the projectors

$$P_V : H \rightarrow H : f \rightarrow \chi_V f \quad , \quad V \in \mathcal{M}$$

as the multiplication operators with the characteristic function. For the spectral measure $E(V) = P_V$, the integrals (1) give the usual position operators of quantum mechanics.

2. For a particle with spin $\frac{1}{2}$, we choose $M = \mathbb{R}^3$, (μ, \mathcal{M}) the Lebesgue measure, and $H = L^2(M) \oplus L^2(M)$ the two-component spinors. We set $P_V(f^\alpha) = \chi_V f^\alpha$ and again define the spectral measure by $E(V) = P_V$.
3. In order to describe a scalar particle whose motion is (by some external forces or constraints) restricted to a submanifold $M \subset \mathbb{R}^3$, we take $H = L^2(M)$, (μ, \mathcal{M}) the measure $\sqrt{g} d^n x$ on M (for the induced Riemannian metric), and $E(V) = P_V$, $P_V(f) = \chi_V f$. For a chart (x^i, U) , the corresponding position operators are given by

$$X^i = \int_U x^i dE_x \quad .$$

In these examples, the vectors of H are functions on the manifold with one or several components. We want to study the question if H can also in the general case be represented as a function space on M . A possible method for this analysis is the functional calculus and constructions similar to the proof of the spectral theorem in its multiplicative form (see e.g. ¹). We proceed in a different way using the notions of “spin scalar product” and “local orthonormal basis,” which is considered to be more transparent.

Definition 2 For $u, v \in H$, we define the complex, bounded measure μ_{uv} by

$$\mu_{uv}(V) = \langle E_V u, v \rangle \quad .$$

Since $\mu_{uv} \ll \mu$, it has a unique Radon-Nikodym representation $d\mu_{uv} = h_{uv} d\mu$ with $h_{uv} \in L^1(M, \mu)$. The mapping

$$\prec, \succ : H \times H \rightarrow L^1(M, \mu) : (u, v) \rightarrow h_{uv}$$

is called **spin scalar product**.

The spin scalar product is linear in the first and anti-linear in the second argument. Furthermore it is positive, $\prec u, u \succ \geq 0$.

Definition 3 A family $(u_l, C_l)_{l \in \mathbb{N}}$ with $u_l \in H$, $C_l \in \mathcal{M}$ is called **local orthonormal basis** (local ONB) if

- (i) $\prec u_l, u_m \succ = \delta_{lm} \chi_{C_l}$
- (ii) The subspace $\langle \{E_V u_l : V \in \mathcal{M}, l \in \mathbb{N}\} \rangle$ is dense in H .

We define a measurable partition $(D_m)_{m \in \mathbb{N} \cup \{0, \infty\}}$ of M by

$$D_m = \{x \in M \mid \#\{l \mid x \in C_l\} = m\} \quad . \quad (2)$$

We say that on D_m the **spin dimension** is m .

Lemma 1 There exists a local orthonormal basis.

Proof. Let $(v_l)_{l \in \mathbb{N}}$ be an orthonormal basis of H . We construct a local ONB (u_l, C_l) in several steps:

1. Using the notation

$$H_u = \overline{\langle \{E_V u : V \in \mathcal{M}\} \rangle} \quad , \quad u \in H \quad , \quad (3)$$

we define the closed subspaces K_l by $K_l = \langle H_{v_1}, \dots, H_{v_l} \rangle$ and construct the series $(w_l)_{l \in \mathbb{N}}$ by $w_1 = v_1$, $w_l = (1 - \text{Pr}_{K_{l-1}}) v_l$ (Pr_K denotes the projector on the closed subspace K). The K_l are a filtration of H , i.e. $K_l \subset K_{l+1}$ and $\bigcup_l K_l = H$. Furthermore, $K_{l-1} \perp w_l \in K_l$ and $K_l = \langle H_{w_1}, \dots, H_{w_l} \rangle$. For $l < m$, $\langle E_V w_l, E_W w_m \rangle = \langle E_{V \cap W} w_l, w_m \rangle = 0$ and thus $H_{w_l} \perp H_{w_m}$. We conclude that $H = \bigoplus_l H_{w_l}$ and $\prec w_l, w_m \succ = 0$ for $l \neq m$.

2. Set $C_l = \{x \mid \prec w_l, w_l \succ(x) \neq 0\}$. We may assume that $\mu(C_l) < \infty$ for all l , because we can otherwise take a partition $(U_i)_{i \in \mathbb{N}} \in \mathcal{M}$ of M with $\mu(U_i) < \infty$ and define the vectors $w_{li} = E_{U_i} w_l$. Then the sets $C_{li} = \{x \mid \prec w_{li}, w_{li} \succ \neq 0\} = C_l \cap U_i$ have finite measure. Since in addition $H_{w_l} = \bigoplus_i H_{w_{li}}$, we can replace the series (w_l) by (w_{li}) .

3. The functions

$$f_l = \frac{\chi_{C_l}}{\sqrt{\prec w_l, w_l \succ}}$$

are in $L^2(M, \mu_{w_l})$, because

$$\int_M |f_l|^2 d\mu_{w_l} = \int_M |f_l|^2 \prec w_l, w_l \succ d\mu = \mu(C_l) < \infty .$$

Acoording to the spectral theorem for unbounded self-adjoint operators, we can thus introduce the vectors u_l by

$$u_l = \left(\int_M f_l(x) dE_x \right) w_l .$$

They satisfy the equation

$$\int_V \prec u_l, u_m \succ d\mu = \int_V d\langle E_x u_l, u_m \rangle = \int_V f_l \overline{f_m} \prec w_l, w_m \succ d\mu .$$

In the case $l \neq m$, we obtain $\prec u_l, u_m \succ = 0$. For $l = m$, we get

$$\int_V \prec u_l, u_l \succ d\mu = \int_V \frac{\chi_{C_l}}{\prec w_l, w_l \succ} \prec w_l, w_l \succ d\mu = \int_V \chi_{C_l} d\mu ,$$

and hence $\prec u_l, u_l \succ = \chi_{C_l}$. We conclude that $\prec u_l, u_m \succ = \delta_{lm} \chi_{C_l}$. Since

$$H_{u_l} = \left\{ \left(\int_M f(x) dE_x \right) u_l , \quad f \in L^2(M, \mu_{u_l}) \right\} = H_{w_l} ,$$

we have $H = \bigoplus_l H_{u_l}$. Thus condition (ii) in Definition 3 is satisfied. \blacksquare

Let in the following (u_l, C_l) be a given local ONB.

Lemma 2 *The functions $\prec v, u_l \succ$ are in $L^2(C_l, \mu)$. The mapping*

$$U : H \rightarrow \bigoplus_l L^2(C_l, \mu) : v \rightarrow \prec v, u_l \succ \quad (4)$$

is unitary and $UE_V U^{-1} = P_V$, where $P_V : f_l \rightarrow f_l \chi_V$ is the multiplication operator with the characteristic function.

Proof. Using the notation (3), Definition 3 implies that

$$H = \bigoplus_l H_{u_l} . \quad (5)$$

We proceed in several steps:

1. We define the operators

$$A_l : \langle \{E_V u_l : V \in \mathcal{M}\} \rangle \rightarrow L^2(C_l, \mu) \quad \text{by} \quad A_l(E_V u_l) = \chi_{(V \cap C_l)} .$$

They are isometric, because

$$\begin{aligned} \langle E_V u_l, E_W u_l \rangle &= \int_M \chi_{(V \cap W)} \langle u_l, u_l \rangle d\mu \\ &= \mu(V \cap W \cap C_l) = \langle \chi_{V \cap C_l}, \chi_{W \cap C_l} \rangle_{L^2(C_l)} . \end{aligned}$$

Since $\mathcal{D}(A_l)$ is dense in H_{u_l} and $\mathcal{R}(A_l)$ is dense in $L^2(C_l)$, the A_l can be uniquely extended to unitary operators $A_l : H_{u_l} \rightarrow L^2(C_l)$. Using the decomposition (5), we define a unitary operator A by

$$A = \bigoplus_l A_l . \quad (6)$$

2. For $V \in \mathcal{M}$ and $l \in \mathbb{N}$, the vector $u := E_v u_l$ satisfies the equation

$$\begin{aligned} A^* P_W A u &= A^* P_W (\chi_{(V \cap C_l)} \delta_{lm})_{m \in \mathbb{N}} = A^* (\chi_{(W \cap V \cap C_l)} \delta_{lm})_{m \in \mathbb{N}} \\ &= E_{W \cap V} u_l = E_W u . \end{aligned}$$

Thus $E_W u = A^* P_W A u$ on $\{E_V u_l : V \in \mathcal{M}\}$. By continuity, this equation also holds on H_{u_l} . Again by continuity and (5), we obtain $E_W = A^* P_W A$.

3. We want to prove that $A = U$. Notice that it is not sufficient to show that $A = U$ on a dense subset of H , because the continuity of U is not obvious. Therefore let $v \in H$ be an arbitrary vector and set $(f_l)_{l \in \mathbb{N}} = Av$. Since f_l vanishes outside C_l , we have

$$\begin{aligned} \int_W (Uv)_l d\mu &= \int_W \langle v, u_l \rangle d\mu = \langle E_W v, u_l \rangle \\ &= \langle E_W A^*(f_m), u_l \rangle = \langle A^* A E_W A^*(f_m), u_l \rangle \\ &= \langle A^* P_W (f_m), u_l \rangle = \langle P_W (f_m), A u_l \rangle_{\bigoplus L^2(C_l)} \\ &= \int_{W \cap C_l} f_l d\mu = \int_W f_l d\mu . \end{aligned}$$

It follows that $(Uv)_l = f_l = (Av)_l$ and hence $U = A$. ■

The unitary operator (4) gives the desired representation of the vectors of H as functions on M . The representation is not unique; it depends on the choice of the local ONB.

The formalism of local ONBs has some analogy with the representation $u = \sum_l \langle u, u_l \rangle u_l$ of a vector in an orthonormal basis (u_l) . As the main difference, the scalar product can be “localized” on operator manifolds with the spectral measure, leading to L^2 -component functions $\langle u, u_l \rangle$ instead of complex coefficients $\langle u, u_l \rangle$. The following corollary extends the formal analogy between ONBs and local ONBs to Parseval’s equations.

Corollary 1 (local completeness relation) For $u, v \in H$,

$$u = \sum_{l=1}^{\infty} \left(\int_{C_l} \langle u, u_l \rangle_x dE_x \right) u_l \quad (7)$$

$$\langle u, v \rangle = \sum_{l=1}^{\infty} \langle u, u_l \rangle \langle u_l, v \rangle \quad a.e. \quad (8)$$

Proof. According to Lemma 2, the function $\langle u, u_l \rangle$ is in $L^2(C_l, \mu) = L^2(C_l, \mu_{u_l u_l})$. We can thus apply the spectral theorem for unbounded, self-adjoint operators and define the vectors

$$w_l = \left(\int_{C_l} \langle u, u_l \rangle_x dE_x \right) u_l \quad .$$

We have $\langle w_l, u_m \rangle = \langle u, u_l \rangle \delta_{lm}$ and thus, with the notation (3), $U w_l = U \Pr_{H_{u_l}} u$. The injectivity of U and (5) yield equation (7).

Since $\langle u, u_l \rangle, \langle v, u_l \rangle \in \bigoplus_l L^2(C_l)$, Lebesgue's monotone convergence theorem yields that

$$\infty > \sum_l \int_M |\langle u, u_l \rangle \langle u_l, v \rangle| d\mu = \int_M \sum_l |\langle u, u_l \rangle \langle u_l, v \rangle| d\mu \quad ,$$

so that the function $f := \sum_l |\langle u, u_l \rangle \langle u_l, v \rangle|$ is in $L^1(M, \mu)$. According to Lemma 2,

$$\begin{aligned} \int_W \langle u, v \rangle d\mu &= \langle E_W u, v \rangle = \langle U E_W U^* U u, U v \rangle_{\bigoplus_l L^2(C_l)} \\ &= \langle P_W U u, U v \rangle = \sum_l \int_W \langle u, u_l \rangle \langle u_l, v \rangle d\mu \quad . \end{aligned} \quad (9)$$

Since the function f dominates the integrand, we can reverse the order of summation and integration in (9) and obtain (8). \blacksquare

We come to the question of how the representation of H as a space of functions on M depends on the choice of the local ONB. We start with a technical lemma.

Lemma 3 Let W be a measurable set with $\mu(W) \neq 0$,

$$U : (L^2(W, \mu))^m \rightarrow (L^2(W, \mu))^n \quad \text{for } m, n \in \mathbb{N} \cup \{0, \infty\}$$

a unitary operator satisfying $U P_V U^{-1} = P_V$ for all $V \in \mathcal{M}$. Then $m = n$.

Proof. We can assume that $\mu(W) < \infty$, because otherwise take a measurable set $V \subset W$ with $0 \neq \mu(V) < \infty$ and consider the restriction of U on $(L^2(V, \mu))^m$. Since the roles of m, n can be interchanged, it suffices to prove $m \leq n$.

Assume that $m > n$. The vectors

$$u^\alpha := (\delta_i^\alpha)_{i=1, \dots, m} \in (L^2(W, \mu))^m \quad , \quad \alpha = 1, \dots, n+1$$

satisfy the equations

$$\langle P_V u^\alpha, u^\beta \rangle = \delta^{\alpha\beta} \mu(V) \quad \text{for } V \subset W \quad .$$

By hypothesis on U , the functions $v^\alpha := U(u^\alpha) = (g_i^\alpha)_{i=1,\dots,n}$ also satisfy

$$\langle P_V v^\alpha, v^\beta \rangle = \delta^{\alpha\beta} \mu(V) \quad \text{for } V \subset W \quad .$$

We thus have for any measurable set $V \subset W$ with $\mu(V) \neq 0$,

$$\delta^{\alpha\beta} = \frac{1}{\mu(V)} \langle P_V v^\alpha, v^\beta \rangle = \frac{1}{\mu(V)} \int_V \sum_{j=1}^n g_j^\alpha \overline{g_j^\beta} d\mu \quad ,$$

and thus

$$\sum_{j=1}^n g_j^\alpha \overline{g_j^\beta} = \delta^{\alpha\beta} \quad \text{a.e.} \quad , \quad \alpha, \beta = 1, \dots, m \quad .$$

This is a contradiction because there are at most n linearly independent vectors in \mathbb{C}^n . ■

Definition 4 Two operator manifolds $(M, \mu, H, E), (\tilde{M}, \tilde{\mu}, \tilde{H}, \tilde{E})$ are called **isomorphic** if there exists a homeomorphism $\phi : M \rightarrow \tilde{M}$ and a unitary operator $U : H \rightarrow \tilde{H}$ such that

(i) ϕ preserves the measure:

$$\mu(V) = \tilde{\mu}(\phi(V)) \quad \text{for all } V \in \mathcal{M}$$

(ii) the spectral measure is invariant:

$$U E_V U^{-1} = \tilde{E}_{\phi(V)} \quad \text{for all } V \in \mathcal{M}$$

In the special case $M = \tilde{M}$ and $\phi = 1$, the operator U satisfies

$$U E_V U^{-1} = \tilde{E}_V \quad \text{for all } V \in \mathcal{M}$$

and is called **isomorphism**.

Theorem 1 Let (M, μ, H, E) be an operator manifold.

1. There exists a partition of M of measurable sets $(D_m)_{m \in \mathbb{N} \cup \{0, \infty\}}$ such that (M, μ, H, E) is isomorphic to the operator manifold

$$(M, \mu, \bigoplus_{m \in \mathbb{N} \cup \{0, \infty\}} (L^2(D_m, \mu))^m, P) \quad ,$$

where P is the canonical spectral measure $P : V \rightarrow P_V$ with $P_V : f_i \rightarrow f_i \chi_V$. The sets D_m are unique (modulo sets of measure zero). They coincide with the partition (D_m) in Definition 3, so that the definition of the spin dimension is independent of the choice of the local ONB.

2. We denote the spin dimension at x by m_x (so $m_x = m$ for $x \in D_m$). For two isomorphisms $V, \tilde{V} : H \rightarrow \bigoplus_{m \in J} (L^2(D_m, \mu))^m$, there are measurable functions W_β^α with

$$\sum_{\alpha=1}^{m_x} W_\beta^\alpha(x) \overline{W_\gamma^\alpha(x)} = \delta_{\beta\gamma} \quad , \quad \sum_{\alpha=1}^{m_x} W_\alpha^\beta(x) \overline{W_\alpha^\gamma(x)} = \delta^{\beta\gamma} \quad \text{for a.a. } x, \quad (10)$$

such that

$$((V\tilde{V}^{-1})f)_x^\alpha = \sum_{\beta=1}^{m_x} W_\beta^\alpha(x) f^\beta(x) \quad . \quad (11)$$

Conversely, if V is an isomorphism and $W_\beta^\alpha(x)$ a family of measurable functions with the property (10), then there is an isomorphism \tilde{V} satisfying (11).

3. Every isomorphism $V : H \rightarrow \bigoplus_m (L^2(D_m, \mu))^m$ can, for a suitable local ONB (u_l, C_l) , be realized as the mapping $\prec \cdot, u_l \succ$.

Proof.

1. We take a local ONB (u_l, C_l) and define the sets D_m by (2). Lemma 2 gives an isomorphism U of (M, μ, H, E) and $(M, \mu, \bigoplus_l L^2(C_l, \mu), P)$. By cutting and recomposing the component functions, we construct a unitary transformation

$$W : \bigoplus_l L^2(C_l, \mu) \rightarrow \bigoplus_{m \in \mathbb{N} \cup \{0, \infty\}} (L^2(D_m, \mu))^m \quad (12)$$

with $WP_V W^{-1} = P_V$. The operator $WU : H \rightarrow \bigoplus_m (L^2(D_m, \mu))^m$ is the desired isomorphism.

In order to show the uniqueness of the sets D_m , we consider two isomorphisms

$$\begin{aligned} V : H &\rightarrow \bigoplus_{m \in \mathbb{N} \cup \{0, \infty\}} (L^2(D_m, \mu))^m \\ \tilde{V} : H &\rightarrow \bigoplus_{m \in \mathbb{N} \cup \{0, \infty\}} (L^2(\tilde{D}_m, \mu))^m \end{aligned}$$

constructed from different local ONBs. Then the mapping

$$\tilde{V}V^{-1} : \bigoplus_m (L^2(D_m, \mu))^m \rightarrow \bigoplus_m (L^2(\tilde{D}_m, \mu))^m$$

is an isomorphism. Assume that $\mu(D_m \cap \tilde{D}_m) \neq \mu(D_m)$ or $\mu(D_m \cap \tilde{D}_m) \neq \mu(\tilde{D}_m)$ for certain m . Since the roles of D_m and \tilde{D}_m can be exchanged, we can assume that there is a set W , $\mu(W) \neq 0$ with $W \subset D_m, W \subset \tilde{D}_n$ and $n \neq m$. Then the restriction of $\tilde{V}V^{-1}$ to $P_W(\bigoplus_{m \in J} (L^2(D_m, \mu))^m)$ is a unitary mapping

$$A : (L^2(W, \mu))^m \rightarrow (L^2(W, \mu))^n$$

with $AP_V A^{-1} = P_V$ for all $V \subset W$. This is a contradiction to Lemma 3. We conclude that D_m, \tilde{D}_m coincide up to sets of measure zero.

2. We set $U = V\tilde{V}^{-1}$. For $u = (f^\alpha), v = (g^\alpha) \in \bigoplus_m (L^2(D_m, \mu))^m$ and $C \in \mathcal{M}$,

$$\int_C \sum_{\alpha=1}^{m_x} f^\alpha \overline{g^\alpha} d\mu = \langle P_C u, v \rangle = \langle P_C Uu, Uv \rangle = \int_C \sum_{\alpha=1}^{m_x} (Uu)^\alpha \overline{(Uv)^\alpha} d\mu$$

(integration and summation can be exchanged according to Lebesgue's dominated convergence theorem). Hence

$$\sum_{\alpha=1}^{m_x} f^\alpha(x) \overline{g^\alpha(x)} = \sum_{\alpha=1}^{m_x} (Uu)^\alpha(x) \overline{(Uv)^\alpha(x)} \quad \text{a.e.} \quad , \quad (13)$$

and we conclude that

$$(Uu)^\alpha(x) = \sum_{\beta=1}^{m_x} U_\beta^\alpha(x) f^\beta(x) \quad \text{a.e.}$$

for suitable coefficients $U_\beta^\alpha(x)$, which are measurable functions in x . The identities (10) follow from (13) and the unitarity of U . Conversely, if some measurable functions $(U_\beta^\alpha(x))_{\alpha,\beta=1,\dots,m_x}$ satisfy (10), we define the isomorphism \tilde{V} by

$$\tilde{V} : u \rightarrow \left(\sum_{\gamma=1}^{m_x} \overline{U_\beta^\gamma(x)} (Vu)^\gamma \right)^\beta .$$

3. Let $V : H \rightarrow \bigoplus_m (L^2(D_m, \mu))^m$ be an isomorphism. We choose a partition $(C_l)_{l \in \mathbb{N}}$ of M subordinate to D_m with $\mu(C_l) < \infty$ and define the mapping $m_l : \mathbb{N} \cup \{0, \infty\} \rightarrow \mathbb{N} \cup \{0, \infty\}$ by the requirement that $m_l = m$ if $C_l \subset D_m$. The vectors

$$(u_{l\alpha})_{l \in \mathbb{N}, \alpha=1,\dots,m_l} := U^{-1} (\chi_{C_l} (\delta^{\alpha\beta}))^\beta$$

satisfy for every $W \in \mathcal{M}$ the relations

$$\begin{aligned} \int_W \langle u_{k\alpha}, u_{l\beta} \rangle d\mu &= \langle E_W u_{k\alpha}, u_{l\beta} \rangle = \langle P_W U u_{k\alpha}, U u_{l\beta} \rangle \\ &= \int_W \chi_{C_k} \chi_{C_l} \delta_{\alpha\beta} d\mu = \delta_{\alpha\beta} \delta_{kl} \mu(C_k \cap W) , \end{aligned}$$

and thus $\langle u_{k\alpha}, u_{l\beta} \rangle = \delta_{\alpha\beta} \delta_{kl} \chi_{C_k}$. Since in addition

$$H_{u_{k\alpha}} = U^{-1} (\iota_\alpha L^2(C_k))$$

(with $\iota_\alpha : L^2(C_k) \hookrightarrow \sum_m L^2(D_m)^m$ the natural injection), it follows that $H = \bigoplus_{l,\alpha} H_{u_{l\alpha}}$. We conclude that $(u_{l\alpha}, C_l)$ is a local ONB. By construction

of $(u_{l\alpha}, C_l)$, we have $V = W \circ (\prec., u_l \succ)_{l \in \mathbb{N}}$, where W is the canonical isomorphism (12). \blacksquare

2. Interpretation, Local Gauge Transformations

In order to clarify the concept of measurability of space, we defined operator manifolds in a general and abstract mathematical setting. For physical applications, one needs to introduce additional objects like the Hamiltonian and the physical wave functions. For our discussion, however, this is not necessary; we prefer to keep the physical interpretation on a general level.

We begin with the case $D_1 = M$ of spin dimension one. According to Theorem 1, H is isomorphic to $L^2(M)$ and can be interpreted as the configuration space of a scalar particle. In contrast to Example 1.2.1, where $H = L^2(M)$ coincided with the function space, H now is only an abstract Hilbert space. The isomorphisms of Theorem 1 give representations of the vectors of H as “wave functions.” The arbitrariness (10),(11) of the representation describes local phase transformations

$$\Psi(x) \longrightarrow e^{i\Lambda(x)} \Psi(x) \quad (14)$$

of the wave functions. This result can be understood qualitatively from the fact that the wave function itself is not observable, only its absolute square $|\Psi(x)|^2$ has a physical interpretation as probability density of the particle. The transformation (14) occurs in quantum mechanics as the local $U(1)$ -gauge transformation of the magnetic field (under which the vector potential behaves like $\vec{A} \rightarrow \vec{A} + (\vec{\nabla}\Lambda)$).

In the case $D_2 = M$ of spin dimension 2, H is isomorphic to $L^2(M) \oplus L^2(M)$ and can be interpreted as two-component Pauli spinors. According to (10),(11), the arbitrariness of the representation as wave functions now describes local $U(2)$ -transformations. These transformations really occur in physics; they correspond to the local $U(2)$ -symmetry in quantum mechanics ³.

The general case allows for the description of m -component wave functions, which is needed for particles with higher spin and for particle multiplets. We again interpret the arbitrariness of the function representation as local gauge freedom:

Definition 5 1. Let (M, H, E) be an operator manifold. An isomorphism

$$V : H \rightarrow \bigoplus_{m \in \mathbb{N} \cup \{0, \infty\}} (L^2(D_m, \mu))^m \quad (15)$$

is called a **gauge**.

2. For two gauges V, \tilde{V} , the mapping

$$U = V\tilde{V}^{-1} : \bigoplus_{m \in \mathbb{N} \cup \{0, \infty\}} (L^2(D_m, \mu))^m \rightarrow \bigoplus_{m \in \mathbb{N} \cup \{0, \infty\}} (L^2(D_m, \mu))^m$$

is called a **gauge transformation**. It can, according to (10),(11), be represented as local $U(m_x)$ -transformation of the wave functions.

The occurrence of local gauge freedom can, on a non-rigorous level, be understood from the fact that only $|\Psi(x)|^2 = \sum_{\alpha=1}^{m_x} |\Psi^\alpha(x)|^2$ is a physical observable. The local gauge group $U(m_x)$ is the isometry group of the spin scalar product.

If taken seriously, our concept has important physical consequences: The local gauge principle is no longer a fundamental physical principle, but follows from the fact that space is a quantum mechanical observable. In contrast to usual gauge theories, the gauge group cannot be chosen arbitrarily. For a given configuration of the spinors, it is fixed to be the group $U(m_x)$, which acts directly on the spinorial index of the wave functions. This is a strong restriction for the formulation of physical models.

We point out that the transformation functions W_β^α in (11) are in general not smooth, they are only measurable. From the physical point of view, it seems reasonable to restrict to smooth gauge transformations. Then the structure of an operator manifold reduces to a principle bundle over M with fibre \mathbb{C}^m and local gauge group $U(m)$. The wave functions are sections of the bundle. In this way, we obtain the mathematical framework of classical gauge theories. In view of the fact that the restriction to smooth gauge transformations is only a technical convenience, however, the question arises if the topology of the fibre bundles has physical significance.

We remark that our constructions have a direct generalization to relativistic quantum mechanics⁴. The adaptation to many-fermion systems finally leads to the “Principle of the Fermionic Projector” as introduced in⁵.

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